This note is intended to be a summary of discussions between myself and Olof Sisask written for the benefit of polymath about how the argument with Bateman on the cap set problem might be generalized to the $Z_N$ case.

The rigor of this will be limited by our still incomplete understanding of Sanders' paper (http://lanl.arxiv.org/abs/1011.0104). Roughly speaking, to begin we need to be able to use Sanders’ paper as a result that is essentially as strong as Meshulam’s result is in the cap set case.

That is, we might begin with $A \subset Z_N$ having density $\frac{1}{(\log N)^{1+\epsilon}}$. In order to avoid keeping track of exponents that are $O(\epsilon)$, we will denote this density by $\frac{1}{(\log N)^{1+\epsilon}}$. But already, we are lying. The set $A$ should really live not in $Z_N$ but in a Bohr set $B$. The Bohr set $B$ should have cardinality at least $\sqrt{N}$ (and be measurably large by other criteria like rank and dimension) and the set $A$ should have density $\frac{1}{(\log N)^{1+\epsilon}}$ inside $B$. All Fourier transforms should be taken relative to $B$. (That is, in the spirit of Lemma 2.8 of Sander’s paper, when we write $\hat{(1_A - \alpha)}$, with $\alpha$ the density of $A$ in $B$, we mean $\hat{\left(\frac{|B|}{N} (1_A - \alpha) \right)}$).

We need to know that with $\alpha \sim \frac{1}{(\log N)^{1+\epsilon}}$, we can bound the Fourier transform above by $\alpha^2$ and that there should be the spectrum $\Delta$, a set of $\alpha^{-3+\epsilon}$ positions where the Fourier transform takes this value. It is not entirely clear to us that Sanders’ result can be interpreted in this way. The restrictions on $B$ need to be made precise; a large Fourier coefficient needs to be shown to give a good increment into an acceptable Bohr set; then everything should follow by something like Sanders’ Lemma 5.2 which gives the iterative procedure by which his theorem follows.

Supposing the above is o.k., it seems we have enough machinery to show that the spectrum $\Delta$ is a set with weak additive structure but without additive smoothing. We have $\alpha^{-3+\epsilon}$ elements of spectrum and they must have $\alpha^{-7+\epsilon}$ additive quadruplets by an argument of Shkedrov. (See here: http://lanl.arxiv.org/abs/math/0605689). Here is what we can do to study the question of additive smoothing. We consider a set $\Lambda \subset \Delta$ with $|\Lambda| = d$. Here $d << \alpha^{-1}$. We would like to study the question of how many elements of $\Delta$ close out $2m$-tuples from $\Lambda$ with $m$ large but depending just on $\epsilon$. That is we study the solutions $\gamma \in \Delta$ to $\gamma = \lambda_1 + \ldots \lambda_m - \lambda_{m+1} - \ldots - \lambda_{2m-1}$, where the $\lambda$’s are taken from $\Lambda$. Now we consider the Bohr set $B' = B(\Lambda, \frac{1}{10m})$. It must be that for any $x \in B'$, we have that $\text{Re}(e(\gamma x)) \geq \frac{1}{2}$). Thus $\gamma$ is in the large spectrum of $B'$ and we may apply Sanders’ Lemma 2.8 which controls the number of elements of $\Delta$ which may lie in the large spectrum of $B'$. The estimate we get is indeed of the form $\alpha^{-1+\epsilon}d^{1+\epsilon}$, the “$Nd^2$”-estimate.

Applying essentially the same probabilistic argument as in my paper with Bateman, we conclude that selections from $\Delta$ of size $\frac{1}{\alpha}^{1+\epsilon}$ have essentially no additive $m$-tuples. This implies additive non-smoothing for the set $\Delta$. At this point, the theory in section 6 of the paper with Bateman applies essentially verbatim, and the output is that we can find a large (density $\frac{1}{\alpha^{0+\epsilon}}$) subset $\Delta' \subset \Delta$ so that $\Delta' \subset \Lambda + H$. 

1
where $\Lambda$ is a “random” set of size $(\frac{1}{\alpha})^{2+}$, which is random in the sense that not more than $d(\frac{1}{\alpha})^{0+}$ elements may belong to the spectrum of a Bohr set like the one defined above and $H$ is a set with $(\frac{1}{\alpha})^{0+}$ additive doubling.

We are not sure about the best way to carry out section 8 of the paper with Bateman, that is to exploit the structure. Freiman seems not too costly, but we can’t quite mod out by G.A.P.’s. We could try using results in this paper of Sanders: http://lanl.arxiv.org/abs/1002.1552, and then mod out by Bohr sets instead. In that case, we still need a good method of removing the low density fibers. However the section 8 argument is quite robust and there is good reason to hope that any sensible approach to it will succeed.

To sum up: if Sanders gives us essentially the numerology of Meshulam, there is a good chance of obtaining the structural information from Bateman-Katz and getting a result on 3-APs.